Homework 1

Math/Stat 561, Algebraic and Geometric Methods in Statistics

Due 18 Jan 2023.

1. (a) Verify the two polynomial equations in Example 1.1.2 of the textbook (Three-step Markov chain) which we discussed in class. That is, fill in all the missing details to verify the equations are a correct characterization of the Markov chain model.

   (b) Does the point $\hat{p} = \left(\frac{20}{78}, \frac{4}{78}, \frac{18}{91}, \frac{3}{91}, \frac{10}{78}, \frac{2}{78}, \frac{24}{91}, \frac{4}{91}\right)$ lie in the 3-step Markov chain model? (This probability distribution shows up on page 7 of the textbook.)

2. Let $M \in \mathbb{R}^{m \times n}$ be an $m \times n$-matrix with entries in $\mathbb{R}$. Prove that the following statements are equivalent:

   (a) The rank of M is at most one.

   (b) There exist vectors $u \in \mathbb{R}^m$ and $v \in \mathbb{R}^n$ such that $M = uv^T$.

   (c) All 2×2-minors of M are zero.

   Advanced version: Instead of proving the above, work on the more general formulation. What analogous statement can you make for 3-dimensional tensors? what is a 'rank' of a tensor?

3. How does the mathematics of problem 2 relate to the model of independence of two discrete random variables?

   Advanced version: Same question for three discrete random variables.

4. Which points in $\mathbb{R}^2$ satisfy

   (a) the equation $p_1^2 + p_2^2 - 1 = 0$?

   (b) the equation $p_1 - p_2^2 = 0$?

   (c) both equations $p_1 - p_2^2 = 0$ and $p_1^2 + p_2^2 - 1 = 0$?

   These solution sets of systems of polynomial equations are *algebraic varieties*. If S is a set of polynomials, then then the variety defined by S is denoted $V(S)$.