Math 561 Assignment 6

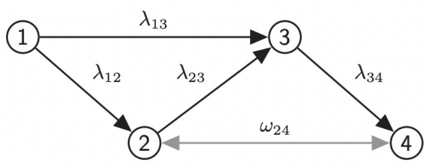
Algebraic and Geometric Methods in Statistics, Spring 2023

Due date: 27 Apr 2023.

Instructions Follow your notes from lectures 21 and 22. (Or the corresponding book sections, as listed in the course lecture schedule.) There are **3 problems**, and an extra credit task at the end.

1. A mixed graph model

Consider the following hidden variable DAG where the hidden confounder influencing random variables X_2 and X_4 has been replaced by the bidirected edge $2 \leftrightarrow 4$. [This procedure is similar to what we did when we introduced the instrumental variable model for hidden variable DAGs.]



In this problem, you will derive the structural equation model for this mixed graph.

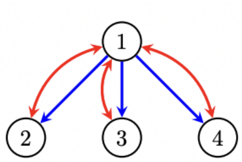
- Complete the linear equations:

$$X_1 = \lambda_{01} + \epsilon_1,$$
$$X_2 = ...$$
$$X_3 = ...$$
$$X_4 = ...$$

- Write the error covariance matrix Ω. For which pairs i and j is $\omega_{ij} = 0$?
- Write the matrices $I_{4\times 4} - \Lambda$ and $(I_{4\times 4} - \Lambda)^{-1}$.
- Compute all entries of the matrix $\Sigma = \begin{bmatrix} \sigma_{11} & \dots & \sigma_{14} \\ \dots & & \\ & & \sigma_{44} \end{bmatrix}$ in two ways:
    a. By comptuing the matrix-equation form of the structrual equation model
    b. By computing the treks between nodes in the mixed graph and verifying that each term in the summation for each σ_{ij} is indeed obtained by enumerating treks in the graph.
- Draw a picture of all the treks between nodes 2 and 4 in the graph.

2. Identifiability

2.a) Consider the 4-node graph in this figure.



The missing bidirected edges give us the followwing equations:

$$\omega_{32} = \sigma_{23} - \lambda_{13}\sigma_{12} - \lambda_{12}(\sigma_{13} - \lambda_{13}\sigma_{11}) = 0,$$

$$\omega_{42} = \sigma_{24} - \lambda_{14}\sigma_{12} - \lambda_{12}(\sigma_{14} - \lambda_{14}\sigma_{11}) = 0,$$

$$\omega_{43} = \sigma_{34} - \lambda_{14}\sigma_{13} - \lambda_{13}(\sigma_{14} - \lambda_{14}\sigma_{11}) = 0.$$

You will now show that the model on this graph is not (linearly) identifiable as we cannot express any subset of $\lambda_{12}, \lambda_{13}, \lambda_{14}$ linearly in terms of Σ:

- Solve the first two equations for λ_{13} and λ_{14}. You should get fractions for both.
- Substitute these into the third equation. You should obtain a quadratic in the indeterminate λ_{12}.
- How many solutions does the quadratic equation have? (Check the discriminant.)

2.b) Prove that if the graph G on n nodes has more than $\binom{n}{2}$ edges, then the model is not (generically) identifiable.

Hint: simply count the number of equations and the number of unknowns. Each missing bidirected edge gives an equation in σ_{uv} and λ_{uv} which says $\omega_{uv} = 0$. Each directed edge corresponds to which unknown? Now, count them.

3. Polynomials

Explore the polynomials that describe this model using a computer (since we haven't talked about doing this by hand!)

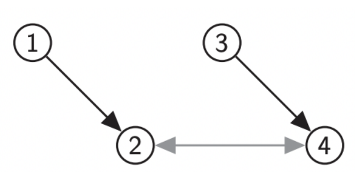
Run the following code in `Macaulay2`:

```
needsPackage "GraphicalModels";
G = mixedGraph(digraph {{1,{2,3}},{2,{3}},{3,{4}}}, bigraph {{2,4}});
R = gaussianRing G;
trekIdeal(R,G)
gaussianVanishingIdeal R
```

- Why did `trekIdeal(R,G)` return zero? This seems to say that there are no trek-separation relations in the graph!
- Study the polynomial that is the output of `gaussianVanishingIdeal R`. Does it look like a determinant? If yes, of what matrix? If no, what does this mean about the model?

[3 - extra credit]

For the polynomial in the last question, consider the following *subgraph* of the graph above:



In this subgraph, verify that there is no trek from node 1 to node 4. Hence, the (1, 4) entry of a certain matrix is zero.

- Which matrix?
- Clear the denominator to derive the polynomial that is the output of `gaussianVanishingIdeal R`.