Log affine models and log linear model ideals, part II

"Algebraic & Geometric Methods in Statistics"

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Recap: Exponential families

- An exponential family is a *parametric statistical model* with probability distributions of a *certain form*.
- General enough to include many of the most common families of probability distributions:
  - multivariate normal
  - exponential
  - Poisson
  - binomial (with fixed number of trials)
- Specific enough to have nice properties:
  - likelihood function is strictly concave [next lecture]
  - have conjugate priors.

Objectives

- What is an exponential family?
- How to find the polynomial ideal of an exponential family?
  - Discrete exponential models: Hypothesis testing [future lecture]
  - Gaussian exponential submodels: Conditional independence implications [past lecture]

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  - What is $\mathcal{A}u$ in this case? $\rightarrow$ $\mathcal{A}u$ is the value of the sufficient statistic in this exponential family. HW: verify that this is sufficient for the binomial model.
  - Can you come up with v such that $Au = Av$?

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  - Can you come up with v such that $Au = Av$? \attn{$\rightarrow v = (1, 0, 1, 0)^t$.
- The corresponding binomial is $p_1 p_3 - p_2^2$.
  - VERIFY that this binomial evaluates to 0 *at all points in the model.*

Example 6.2.6. - class & board work

The model of independence of two discrete random variables. Say that $r_1 = 4$ and $r_2 = 3$.

- What is the parametrization of the model?
- What is the design matrix $\mathcal{A}$?
- From an observed table of counts u (which format is the table in, by the way??), what does Au compute?
- Find some generators of the toric ideal I_A by hand. Interpet them.
- How do you know when you have *all* binomials that suffice to capture (generate) the entire ideal of the model?
    - . . . That's the million dollar question!

Self-study

- We leave 6.2.7 for self-study and reading at your own pace.
  - This is good/useful for homework 2.
- You should try the following `Macaulay2` code for computing ideal generators of $I_{\mathcal{A}}$ from the matrix $\mathcal{A}$ – see next slide.
  - Try it on your examples as well as the class examples.

```
14 : loadPackage "FourTiTwo";
i15 : A = matrix"3,2,1,0;0,1,2,3"
o15 = | 3 2 1 0 |
      | 0 1 2 3 |
               2        4
o15 : Matrix ZZ  <--- ZZ
i16 : toricMarkov A -- I_A generators: vector format
o16 = | 0 1  -2 1 |
      | 1 -2 1  0 |
      | 1 -1 -1 1 |
               3        4
o16 : Matrix ZZ  <--- ZZ
i17 : R=QQ[p_1,p_2,p_3,p_4];
i18 : toricMarkov(A,R) -- I_A generators: polynomial format
                2            2
o18 = ideal (- p  + p p , - p  + p p , - p p  + p p )
                3    2 4     2    1 3     2 3    1 4
o18 : Ideal of R
```

Next up:

How to use implicit models for likelihood inference.

- next topic: likelihood inference from ch7.

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