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- Form the homotopy system $H(x,t) = (1-t) \cdot F(x) + t \cdot G(x)$
- Use predictor-corrector methods to track the numerical solutions as t moves from $t = 1$ to $t = 0$.

Homotopy Tracking

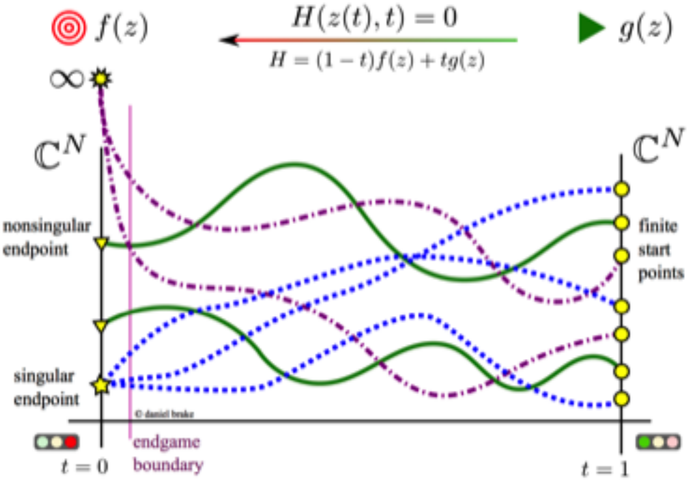

Figure: Homotopy Continuation Illustration (Dani Brake)

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Proof Sketch

- By *Birch's Theorem*, a homotopy between the two systems is given by

$$H(\theta, t) := t\left(A\hat{p}_{stat} - \frac{1}{N}Au\right) + (1-t)\left(A\hat{p}_{win} - \frac{1}{N}Au\right)$$

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- Left to show tracking paths do not intersect (we show the Jacobian matrix of the system has always full rank)

□

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- Knowing how scaling vectors c affect the ML degree of a particular toric model V_A allows us to find a convenient c_{win} (e.g. such that the model has *low* ML degree).
- By the Theorem, we can now find the MLE $\hat{\theta}_{win}$ and track its unique homotopy path to find the original MLE of interest $\hat{\theta}_{stat}$.

Application Example

Example (Veronese revisited)

Recall
$$A = \begin{bmatrix} 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 2 \end{bmatrix},$$
with $u = (1, 3, 5, 7, 9, 2)$. Here $c_{stat} = (1, 1, 1, 1, 1, 1)$.

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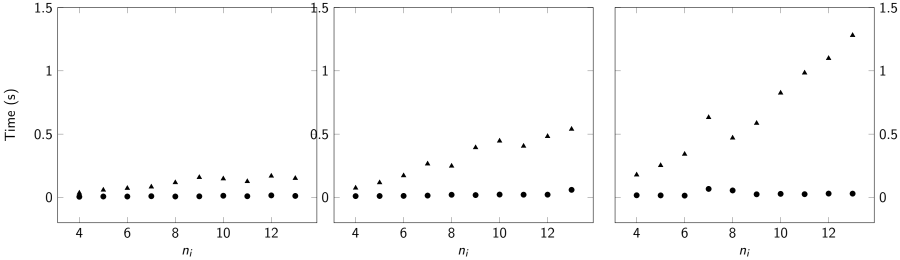
Success story



$d - 1 = 5$ $\quad$ $d - 1 = 10$ $\quad$ $d - 1 = 15$

Figure: Running times of iterative proportional scaling (triangles) versus path tracking (circles) on rational normal scrolls. Average of 7 trials.