

Worksheet 2

Math/Stat 561, Algebraic and Geometric Methods in Statistics

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Group members: Write your names here.

1 Conditional independence ideals

Definition

Proposition (4.1.6.) & Definition (4.1.7.): If X is a discrete random vector $X = (X_1, \dots, X_m)$, then the CI statement $X_A \perp\!\!\!\perp X_B | X_C$ is equivalent to

$$p_{i_A, i_B, i_C, +} \cdot p_{j_A, j_B, i_C, +} - p_{i_A, j_B, i_C, +} \cdot p_{j_A, i_B, i_C, +} = 0$$

for all possible states of the variables i_A, j_A, i_B, j_B , and i_C .

The CI ideal $I_{A \perp\!\!\!\perp B | C}$ is the set of polynomials generated by all quadratic polynomials above.

Task

Verify that the following polynomials are the correct polynomials for the ideal of the statement $gender \perp\!\!\!\perp hair | soccer$ from lecture 4.

$$\begin{aligned} & -p_{1,2,1,1}p_{2,1,1,1} - p_{1,2,1,2}p_{2,1,1,1} - p_{1,2,1,1}p_{2,1,1,2} - p_{1,2,1,2}p_{2,1,1,2} + p_{1,1,1,1}p_{2,2,1,1} + \\ & p_{1,1,1,2}p_{2,2,1,1} + p_{1,1,1,1}p_{2,2,1,2} + p_{1,1,1,2}p_{2,2,1,2}, \\ & -p_{1,2,1,1}p_{3,1,1,1} - p_{1,2,1,2}p_{3,1,1,1} - p_{1,2,1,1}p_{3,1,1,2} - p_{1,2,1,2}p_{3,1,1,2} + p_{1,1,1,1}p_{3,2,1,1} + \\ & p_{1,1,1,2}p_{3,2,1,1} + p_{1,1,1,1}p_{3,2,1,2} + p_{1,1,1,2}p_{3,2,1,2}, \\ & -p_{2,2,1,1}p_{3,1,1,1} - p_{2,2,1,2}p_{3,1,1,1} - p_{2,2,1,1}p_{3,1,1,2} - p_{2,2,1,2}p_{3,1,1,2} + p_{2,1,1,1}p_{3,2,1,1} + \\ & p_{2,1,1,2}p_{3,2,1,1} + p_{2,1,1,1}p_{3,2,1,2} + p_{2,1,1,2}p_{3,2,1,2}, \\ & -p_{1,2,2,1}p_{2,1,2,1} - p_{1,2,2,2}p_{2,1,2,1} - p_{1,2,2,1}p_{2,1,2,2} - p_{1,2,2,2}p_{2,1,2,2} + p_{1,1,2,1}p_{2,2,2,1} + \\ & p_{1,1,2,2}p_{2,2,2,1} + p_{1,1,2,1}p_{2,2,2,2} + p_{1,1,2,2}p_{2,2,2,2}, \\ & -p_{1,2,2,1}p_{3,1,2,1} - p_{1,2,2,2}p_{3,1,2,1} - p_{1,2,2,1}p_{3,1,2,2} - p_{1,2,2,2}p_{3,1,2,2} + p_{1,1,2,1}p_{3,2,2,1} + \\ & p_{1,1,2,2}p_{3,2,2,1} + p_{1,1,2,1}p_{3,2,2,2} + p_{1,1,2,2}p_{3,2,2,2}, \\ & -p_{2,2,2,1}p_{3,1,2,1} - p_{2,2,2,2}p_{3,1,2,1} - p_{2,2,2,1}p_{3,1,2,2} - p_{2,2,2,2}p_{3,1,2,2} + p_{2,1,2,1}p_{3,2,2,1} + \\ & p_{2,1,2,2}p_{3,2,2,1} + p_{2,1,2,1}p_{3,2,2,2} + p_{2,1,2,2}p_{3,2,2,2}. \end{aligned}$$

2 Explicit points in the 3-step Markov chain model

Recall Example 1.1.2 from the book: 3-step Markov chain.

$$p_{ijk} = P(X_1 = i, X_2 = j, X_3 = k) \text{ and } P(X_3 = k | X_1 = i, X_2 = j) = \frac{p_{ijk}}{p_{ij+}}.$$

You verified that a probability distribution, represented by a vector of probabilities $p = (p_{000}, p_{001}, p_{010}, p_{011}, p_{100}, p_{101}, p_{110}, p_{111}) \in \mathbb{R}^8$, being in this model is *equivalent* to the following four conditions:

$$p_{ijk} \geq 0 \text{ for all } i, j, k \in \{0, 1\}, \quad \sum_{i,j,k} p_{ijk} = 1,$$

$$p_{000}p_{101} - p_{001}p_{100} = 0, \text{ and } p_{010}p_{111} - p_{011}p_{110} = 0.$$

1. In this example: **what is the variety?**
2. Is the point $(1/8, 1/8, \dots, 1/8)$ **on** this variety? (That is, is this joint probability vector in the model?)
3. Find an example of a point on the variety, which is a point in this model.