Worksheet 2

Math/Stat 561, Algebraic and Geometric Methods in Statistics

23 January 2023

Group members: Write your names here.

1 Conditional independence ideals

Definition

Proposition (4.1.6.) & Definition (4.1.7.): If X is a discrete random vector $X = (X_1, \ldots, X_m)$, then the CI statement $X_A \perp\!\!\!\perp X_B | X_C$ is equivalent to

$$p_{i_A,i_B,i_C,+} \cdot p_{j_A,j_B,i_C,+} - p_{i_A,j_B,i_C,+} \cdot p_{j_A,i_B,i_C,+} = 0$$

for all possible states of the variables i_A, j_A, i_B, j_B, and i_C.

The *CI ideal* $I_{A \perp\!\!\!\perp B|C}$ is the set of polynomials *generated by all* quadratic polynomials above.

Task

Verify that the following polynomials are the correct polynomials for the ideal of the statement *gender* $\perp\!\!\!\perp$ *hair*|*soccer* from lecture 4.

$-p_{1,2,1,1}p_{2,1,1,1} - p_{1,2,1,2}p_{2,1,1,1} - p_{1,2,1,1}p_{2,1,1,2} - p_{1,2,1,2}p_{2,1,1,2} + p_{1,1,1,1}p_{2,2,1,1} + p_{1,1,1,2}p_{2,2,1,1} + p_{1,1,1,1}p_{2,2,1,2} + p_{1,1,1,2}p_{2,2,1,2}$,

$-p_{1,2,1,1}p_{3,1,1,1} - p_{1,2,1,2}p_{3,1,1,1} - p_{1,2,1,1}p_{3,1,1,2} - p_{1,2,1,2}p_{3,1,1,2} + p_{1,1,1,1}p_{3,2,1,1} + p_{1,1,1,2}p_{3,2,1,1} + p_{1,1,1,1}p_{3,2,1,2} + p_{1,1,1,2}p_{3,2,1,2}$,

$-p_{2,2,1,1}p_{3,1,1,1} - p_{2,2,1,2}p_{3,1,1,1} - p_{2,2,1,1}p_{3,1,1,2} - p_{2,2,1,2}p_{3,1,1,2} + p_{2,1,1,1}p_{3,2,1,1} + p_{2,1,1,2}p_{3,2,1,1} + p_{2,1,1,1}p_{3,2,1,2} + p_{2,1,1,2}p_{3,2,1,2}$,

$-p_{1,2,2,1}p_{2,1,2,1} - p_{1,2,2,2}p_{2,1,2,1} - p_{1,2,2,1}p_{2,1,2,2} - p_{1,2,2,2}p_{2,1,2,2} + p_{1,1,2,1}p_{2,2,2,1} + p_{1,1,2,2}p_{2,2,2,1} + p_{1,1,2,1}p_{2,2,2,2} + p_{1,1,2,2}p_{2,2,2,2}$,

$-p_{1,2,2,1}p_{3,1,2,1} - p_{1,2,2,2}p_{3,1,2,1} - p_{1,2,2,1}p_{3,1,2,2} - p_{1,2,2,2}p_{3,1,2,2} + p_{1,1,2,1}p_{3,2,2,1} + p_{1,1,2,2}p_{3,2,2,1} + p_{1,1,2,1}p_{3,2,2,2} + p_{1,1,2,2}p_{3,2,2,2}$,

$-p_{2,2,2,1}p_{3,1,2,1} - p_{2,2,2,2}p_{3,1,2,1} - p_{2,2,2,1}p_{3,1,2,2} - p_{2,2,2,2}p_{3,1,2,2} + p_{2,1,2,1}p_{3,2,2,1} + p_{2,1,2,2}p_{3,2,2,1} + p_{2,1,2,1}p_{3,2,2,2} + p_{2,1,2,2}p_{3,2,2,2}$.

2 Explicit points in the 3-step Markov chain model

Recall Example 1.1.2 from the book: 3-step Markov chain.

$p_{ijk} = P(X_1 = i, X_2 = j, X_3 = k)$ and $P(X_3 = k | X_1 = i, X_2 = j) = \frac{p_{ijk}}{p_{ij+}}$.

You verified that a probability distribution, represented by a vector of probabilities $p = (p_{000}, p_{001}, p_{010}, p_{011}, p_{100}, p_{101}, p_{110}, p_{111}) \in \mathbb{R}^8$, being in this model is *equivalent to* the following four conditions:

$$p_{ijk} \geq 0 \text{ for all } i, j, k \in \{0, 1\}, \qquad \sum_{i,j,k} p_{ijk} = 1,$$

$$p_{000}p_{101} - p_{001}p_{100} = 0, \text{ and } p_{010}p_{111} - p_{011}p_{110} = 0.$$

1. In this example: **what is the variety**?
2. Is the point $(1/8, 1/8, \ldots, 1/8)$ **on** this variety? (That is, is this joint probability vector in the model?)
3. Find an example of a point on the variety, which is a point in this model.