Math 561 Worksheet 3

Group members:

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1. The design matrix and log-linear models

Consider the model $p_{ij} = \alpha_i \beta_j$ for $i \in [2]$ and $j \in [2]$. This is the model of binary independent random variables. The design matrix is $A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$. What is the analogous matrix B for the model of independence of two random variables with $X_1 \in [2]$ and $X_2 \in [3]$?

2. Rowspan and points in the model

- Give an example of an element in the vector space $rowspan(B)$ for the matrix B in the previous problem.
- Pick a value of the parameter vector $[\alpha_1, \alpha_2, \beta_1, \beta_2, \beta_3]$ and write the corresponding point $p = [p_{11}, p_{12}, p_{13}, p_{21}, p_{22}, p_{23}]$ in the model $\mathcal{M}_{X_1 \perp\!\!\!\perp X_2}$.
- Verify whether $\log p$, for this particular point p, is the the row span of the design matrix B for the model.

3. MLE for a log-linear model

Continue exploring the model $\mathcal{M}_{X_1 \perp\!\!\!\perp X_2}$ with X_1 binary and X_2 ternary.

Suppose the observed data count table is:

$$u = \begin{bmatrix} 1 & 10 & 100 \\ 100 & 10 & 1 \end{bmatrix}.$$

- What is the MLE for this data table u?
- Does the MLE belong in the model $\mathcal{M}_{X_1 \perp\!\!\!\perp X_2}$? How certain are you of your answer?

4. Toric ideal of log-linear models

Continue exploring the model $\mathcal{M}_{X_1 \perp\!\!\!\perp X_2}$ with X_1 binary and X_2 ternary, with the design matrix B you derived in problem #1.

- Give an example of a binomial in the ideal I_B of this model.
- What is the value of the sufficient statistic vector for the data point u from problem #3?
- Can you find another synthetic data table, v, whose sufficient statistic *[Hint: Bv]* has the same value?
- What is the explicit form of the monomials p^u and p^v in this example?
- Is it true that $p^u - p^v \in I_B$?
  - Can you verify this on a computer?